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Determination of vector constant of motion for a particle moving in a conservative central force field

C C Yan

Instituto de Física, Universidade Federale do Rio de Janeiro, Ilha do Fundão, Rio de Janeiro, RJ, Brazil

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Abstract. A simple method is proposed to determine vector constants of motion for a particle moving in a conservative central force field.

1. Introduction

According to the general theory the motion of a particle in a force field has six constants of motion [1]. The knowledge of all the constants of the motion is tantamount to the knowledge of the complete solutions. If a sufficient number of scalar constants of motion are available one can certainly define a vector constant of motion by combination of these scalar constants of the motion. Thus the existence of some vector constant of motion is to be expected, rather than to be regarded as some unusual surprise, for any integrable problem. The trouble is that constants of motion cannot be easily obtained in general. For the simple case of conservative central force field there are some trivial constants of motion. One is a scalar quantity representing the total energy of the particle. The other is an axial quantity representing the angular momentum of the particle with respect to the centre of the force. However, even for the simple case of a conservative central force field, other constants of motion do not come easily. Thus the discovery of the Laplace–Runge–Lenz vector [2–7] as a vector constant of motion for the Kepler–Coulomb problem is regarded as a peculiar consequence of the inverse square force law. Though the force law that governs the motion of an isotropic harmonic oscillator is just as simple as or simpler than the inverse square law, it is a symmetric tensor rather than a vector that is well known to be a constant of motion for this case [7–9]. This fact seems to lead to the idea that the existence of a vector constant of motion depends very much on the particular form of the force law with some hidden higher symmetry. It is true that the explicit form of a vector constant of motion does and should depend on the form of the force law. But the existence of a vector constant of motion, according to the general theory, should be expected and should not depend on the form of the force law as long as it is integrable. Thus it is natural to search for similar vector constants of motion for force laws other than the Kepler–Coulomb problem [10–16].

It is well known that a knowledge of a vector constant of motion leads to the determination of the equation of the orbit. One may thus expect that a knowledge of the equation of the orbit may help to determine the vector constant of motion. Now the problems of the orbits for the conservative central force case can always be solved

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by quadratures [1, 7]. One may suspect that vector constants of motion can also be determined in terms of quadratures. It is our purpose here to demonstrate explicitly, using a simple method, how to determine vector constants of motion.

2. Basic formulation for the determination of vector constants of motion

The equation of motion for a particle of unit mass moving in a conservative central force field can be written in the following way

$$\ddot{\boldsymbol{r}}=f(x)\boldsymbol{r} \tag{2.1}$$

where

$$\dot{\boldsymbol{r}}\equiv\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{r} \tag{2.2}$$

$$x\equiv\boldsymbol{r}\cdot\boldsymbol{r}=r^2. \tag{2.3}$$

It is well known that there are two obvious constants of motion for (2.1). One is the angular momentum given by

$$\boldsymbol{L}=\boldsymbol{r}\times\dot{\boldsymbol{r}} \tag{2.4}$$

and the other is the total energy given by

$$H=\tfrac{1}{2}\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}}-\tfrac{1}{2}\int^x f(s)\,\mathrm{d}s. \tag{2.5}$$

From (2.4) and (2.5) we obtain

$$F(x)\equiv(\boldsymbol{r}\cdot\dot{\boldsymbol{r}})^2=2Hx+x\int^x f(s)\,\mathrm{d}s-L^2. \tag{2.6}$$

Now if $\boldsymbol{J}$ is a vector constant of motion independent of $\boldsymbol{L}$ it may be assumed to lie on the plane perpendicular to $\boldsymbol{L}$. Thus it may be assumed to have the following form

$$\boldsymbol{J}=A\dot{\boldsymbol{r}}+B\boldsymbol{r} \tag{2.7}$$

where A and B are some scalar functions to be determined so that $\dot{\boldsymbol{J}}=0$. Taking time derivative of (2.7) and comparing with (2.1) we get

$$\dot{A}+B=0 \tag{2.8}$$

$$\dot{B}+Af=0. \tag{2.9}$$

The determination of a vector constant of motion is thereby transformed to the determination of the solution to equations (2.8) and (2.9).

Combining (2.8) and (2.9) we have

$$\ddot{A}=fA. \tag{2.10}$$

Although A and B may in general depend on $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$ the dependence on $\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}}$ and $\boldsymbol{r}\cdot\dot{\boldsymbol{r}}$ etc may be expressed in terms of x by using (2.5) and (2.6). Hence without loss of generality one may assume that A and B are functions of x only. Now noting that

$$\dot{x}=2F^{1/2} \tag{2.11}$$

$$(\dot{F}^{1/2})=\tfrac{1}{2}F^{-1/2}\left(\frac{\mathrm{d}}{\mathrm{d}x}F\right)\dot{x}=\frac{\mathrm{d}}{\mathrm{d}x}F \tag{2.12}$$

we have

$$\dot{A}=\left(\frac{\mathrm{d}}{\mathrm{d}x}A\right)\dot{x}=2F^{1/2}\frac{\mathrm{d}}{\mathrm{d}x}A \tag{2.13}$$

$$\ddot{A}=4F\frac{\mathrm{d}^2}{\mathrm{d}x^2}A+2\left(\frac{\mathrm{d}}{\mathrm{d}x}F\right)\frac{\mathrm{d}}{\mathrm{d}x}A. \tag{2.14}$$

Thus equation (2.10) becomes

$$4F\frac{\mathrm{d}^2}{\mathrm{d}x^2}A+2\left(\frac{\mathrm{d}}{\mathrm{d}x}F\right)\frac{\mathrm{d}}{\mathrm{d}x}A=fA \tag{2.15}$$

which can also be written as

$$4\frac{\mathrm{d}}{\mathrm{d}x}\left(F^{1/2}\frac{\mathrm{d}}{\mathrm{d}x}A\right)=f(F^{-1/2}A). \tag{2.16}$$

Now introducing W defined by

$$F^{1/2}\frac{\mathrm{d}}{\mathrm{d}x}A=AW \tag{2.17}$$

one can transform (2.15) into the following form

$$F^{1/2}\frac{\mathrm{d}}{\mathrm{d}x}W+W^2=\tfrac{1}{4}f. \tag{2.18}$$

Furthermore from (2.8) we have

$$B=-\dot{A}=-\left(\frac{\mathrm{d}}{\mathrm{d}x}A\right)\dot{x}=-2F^{1/2}\frac{\mathrm{d}}{\mathrm{d}x}A. \tag{2.19}$$

Hence from (2.7) and (2.19) we obtain

$$\boldsymbol{J}\cdot\boldsymbol{r}=AF^{1/2}+Bx=F^{1/2}\left[A-2x\frac{\mathrm{d}}{\mathrm{d}x}A\right]. \tag{2.20}$$

When the equation of orbit is not yet found one can utilize (2.15), (2.16) or (2.18) to determine A and B so as to lead to the determination of $\boldsymbol{J}$. On the other hand when the equation of orbit is already known one can use (2.20) to determine A and, from such a result, to arrive at the determination of $\boldsymbol{J}$.

3. Simple examples

3.1. Kepler–Coulomb problem

For this problem we have

$$f(x)=-kx^{-3/2}. \tag{3.1}$$

Hence

$$F(x)=2Hx+2kx^{1/2}-L^2 \tag{3.2}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}F(x)=x\frac{\mathrm{d}}{\mathrm{d}x}f+2f=-\tfrac{1}{2}kx^{-3/2}=\tfrac{1}{2}f. \tag{3.3}$$

Thus by inspection one finds that

$$A = F^{1/2} \tag{3.4}$$

is a solution to (2.16). From (2.8) and (3.4) one obtains

$$B = -\dot{A} = -\frac{\mathrm{d}}{\mathrm{d}x} F = -\left[2H + \frac{k}{r}\right]. \tag{3.5}$$

Therefore

$$\boldsymbol{J} = (\boldsymbol{r} \cdot \dot{\boldsymbol{r}})\dot{\boldsymbol{r}} - \left[2H + \frac{k}{r}\right]\boldsymbol{r} = \boldsymbol{L} \times \dot{\boldsymbol{r}} + \frac{k}{r}\boldsymbol{r}. \tag{3.6}$$

This is nothing but the well known Laplace–Runge–Lenz vector.

Taking the inner product of $\boldsymbol{J}$ and $\boldsymbol{r}$ we have

$$\boldsymbol{J} \cdot \boldsymbol{r} = Jr \cos\theta = (\boldsymbol{r} \cdot \dot{\boldsymbol{r}})^2 - \left(2H - \frac{k}{r}\right) r^2 = kr - L^2. \tag{3.7}$$

Thus

$$\frac{1}{r} = \frac{k}{L^2}\left(1 - \frac{J}{k}\cos\theta\right). \tag{3.8}$$

This is the well known equation of the orbit for this problem [1, 7] where the eccentricity e can be identified as J/k. This shows that the knowledge of a vector constant of motion $\boldsymbol{J}$ does lead to the determination of the equation of the orbit.

3.2. Isotropic harmonic oscillator

For this problem it is a symmetric tensor rather than a vector that is better known to be a constant of the motion [7–9]. However by applying our simple method a vector constant of motion can be easily found. We have

$$f = -k \tag{3.9}$$

$$F = 2Hx - kx^2 - L^2. \tag{3.10}$$

Hence (2.18) becomes

$$[W^2 + (k/4)]^{-1} \frac{\mathrm{d}W}{\mathrm{d}x} = -[2Hx - kx^2 - L^2]^{-1/2}. \tag{3.11}$$

Integrating (3.11) we obtain

$$2\tan^{-1}\left(\frac{2}{\sqrt{k}} W\right) = \sin^{-1}\left[\frac{(H - kx)}{\sqrt{H^2 - kL^2}}\right] + 2\alpha \tag{3.12}$$

where α is an integration constant and

$$H^2 - kL^2 > 0 \tag{3.13}$$

$$\sqrt{H^2 - kL^2} > |H - kx|. \tag{3.14}$$

Solving for W from (3.12) we get

$$W=\frac{\sqrt{k}}{2}\tan\phi \tag{3.15}$$

$$\phi=\tfrac{1}{2}\sin^{-1}y+\alpha \tag{3.16}$$

$$y=\frac{H-kx}{\sqrt{H^2-kL^2}}. \tag{3.17}$$

Combining (2.17) and (3.15) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln A=\frac{\mathrm{d}}{\mathrm{d}x}\ln[\cos\phi]. \tag{3.18}$$

Hence from (3.18), by setting $\alpha=\pi/4$ for later convenience, we have

$$\begin{aligned}A=\beta\cos\phi&=\frac{\beta}{\sqrt{2}}\left\{\left[1+\sqrt{\frac{kF}{H^2-kL^2}}\right]^{1/2}-\left[1-\sqrt{\frac{kF}{H^2-kL^2}}\right]^{1/2}\right\}\\&=\beta[H^2-kL^2]^{-1/4}[\sqrt{H^2-kL^2}-H+kx]^{1/2}\end{aligned} \tag{3.19}$$

where β is an integration constant. From (2.19) and (3.19) we obtain

$$B=-\beta\sqrt{k}[H^2-kL^2]^{-1/4}[\sqrt{H^2-kL^2}+H-kx]^{1/2}. \tag{3.20}$$

Thus the sought for vector constant of motion is given by

$$\boldsymbol{J}=\beta[H^2-kL^2]^{-1/4}\{[\sqrt{H^2-kL^2}-H+kx]^{1/2}\dot{\boldsymbol{r}}-\sqrt{k}[\sqrt{H^2-kL^2}+H-kx]^{1/2}\boldsymbol{r}\} \tag{3.21}$$

$$J^2=2\beta^2[H-\sqrt{H^2-kL^2}]. \tag{3.22}$$

Taking the inner product of (3.21) with $\boldsymbol{r}$ we obtain

$$\boldsymbol{J}\cdot\boldsymbol{r}=Jr\cos\theta'=\frac{-1}{k}[H-\sqrt{H^2-kL^2}]B. \tag{3.23}$$

Comparing with the equation of orbit given by Whittaker [1]

$$\frac{1}{r^2}=\frac{c}{2}+\left(\frac{c^2}{4}-\frac{k}{L^2}\right)^{1/2}\cos(2\theta-2\gamma) \tag{3.24}$$

we find the relations

$$\theta'=\theta-\gamma \tag{3.25}$$

$$\frac{2H}{L^2}=c. \tag{3.26}$$

A second vector constant of motion on the orbit plane can be obtained as

$$\boldsymbol{J}_2=\frac{1}{kL^2}[H+\sqrt{H^2-kL^2}]\boldsymbol{L}\times\boldsymbol{J}=-\frac{1}{k}B\dot{\boldsymbol{r}}+A\boldsymbol{r}. \tag{3.27}$$

Introducing a dyadic defined by

$$\mathbf{M}=\boldsymbol{J}\boldsymbol{J}+k\boldsymbol{J}_2\boldsymbol{J}_2=2\beta^2[\dot{\boldsymbol{r}}\dot{\boldsymbol{r}}+k\boldsymbol{r}\boldsymbol{r}] \tag{3.28}$$

one can see that the symmetric tensor generated by this dyadic is equal to the well known symmetric tensor constant of the motion for this problem [7-9].

3.3. *Inverse cube force law*

For this case we have

$$f = -kx^{-2} \tag{3.29}$$

$$F = 2Hx - (L^2 - k). \tag{3.30}$$

Thus equation (2.15) becomes

$$4[2Hx - (L^2 - k)]\frac{\mathrm{d}^2}{\mathrm{d}x^2}A + 4H\frac{\mathrm{d}}{\mathrm{d}x}A + kx^{-2}A = 0. \tag{3.31}$$

Now assuming that $L^2 - k > 0$ and introducing the following transformation

$$z = (L^2 - k)/2Hx \tag{3.32}$$

one obtains from (3.31)

$$z(1-z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}A + (\tfrac{3}{2} - 2z)\frac{\mathrm{d}}{\mathrm{d}z}A + \frac{k}{4(L^2 - k)}A = 0 \tag{3.33}$$

which is a hypergeometric equation [17] with

$$a = \frac{1+n}{2} \qquad b = \frac{1-n}{2} \qquad c = \tfrac{3}{2} \qquad n = \frac{L}{\sqrt{L^2 - k}}. \tag{3.34}$$

Hence the solution is [17]

$$A = {}_2F_1\left[\frac{1+n}{2}, \frac{1-2}{2}; \frac{3}{2}, z\right] = \frac{\sin[n \sin^{-1}(\sqrt{z})]}{n\sqrt{z}} \tag{3.35}$$

$$B = \frac{2H}{\sqrt{L^2 - k}}\sqrt{z}\cos[n \sin^{-1}(\sqrt{z})] - \frac{2H}{L}\sqrt{1-z}\sin[n \sin^{-1}(\sqrt{z})]. \tag{3.36}$$

The equation of orbit is given by

$$\boldsymbol{J} \cdot \boldsymbol{r} = Jr\cos\theta = AF^{1/2} + Bx = \left[\frac{L^2 - k}{z}\right]^{1/2}\cos[n \sin^{-1}(\sqrt{z})]. \tag{3.37}$$

Comparing with the equation of orbit called Cortes' spiral given by Whittaker [1]

$$\frac{1}{r} = \mathscr{A}\cos\left[\frac{1}{n}\theta + \varepsilon\right] \tag{3.38}$$

we arrive at the following identities

$$\varepsilon = -\frac{\pi}{2} \tag{3.39}$$

$$\mathscr{A} = \frac{2H}{L^2 - k} \tag{3.40}$$

$$J = \sqrt{2H}. \tag{3.41}$$

4. The general solution

The three simple examples discussed in the last section correspond to the three cases that the equations of orbit can be expressed in terms of circular functions [1, 7]. Now the equation of orbit for a conservative central force field can in general be expressed as [1, 7]

$$\theta = \int^r \left[\frac{2H}{L^2} + \frac{1}{L^2}\int^{r^2} f(s)\,\mathrm{d}s - \frac{1}{r^2}\right]^{-1/2} \frac{\mathrm{d}r}{r^2} = \frac{L}{2}\int^x F^{-1/2}\frac{\mathrm{d}s}{s}. \tag{4.1}$$

Thus from (2.20) and (4.1) we obtain the general solution

$$A = -J\frac{\sqrt{x}}{2}\int^x F^{-1/2}\cos\theta\frac{\mathrm{d}s}{s} = \frac{J}{L}\sqrt{x}\sin\left[\frac{L}{2}\int^x F^{-1/2}\frac{\mathrm{d}s}{s}\right]. \tag{4.2}$$

It can be directly verified that (4.2) is indeed a solution to (2.15) or (2.16). Using (2.19) and (4.2) we obtain

$$B = -\frac{J}{L}F^{1/2}\frac{1}{\sqrt{x}}\sin\left[\frac{L}{2}\int^x F^{-1/2}\frac{\mathrm{d}s}{s}\right] - J\frac{1}{\sqrt{x}}\cos\left[\frac{L}{2}\int^x F^{-1/2}\frac{\mathrm{d}s}{s}\right]. \tag{4.3}$$

Expression (2.7) with A and B given by (4.2) and (4.3) is a vector constant of the motion for any given conservative central force field f. Therefore it is seen that in any conservative central force field there is a vector constant motion on the plane of the orbit. Furthermore by taking the vector product of $\boldsymbol{L}$ with such a vector constant of motion another vector constant of motion lying on the orbit plane can be obtained. From this view point there is nothing special about the Laplace–Runge–Lenz vector except that its structure is rather simple.

Although the expressions (4.2) and (4.3) do not in general yield simple results expressible by elementary functions they can nevertheless be utilized to find at least two more vector constants of motion that can be expressed in terms of elementary functions. They are namely (1) the force law given by $f = -kx^{-3/2} + \mu x^{-2}$ and (2) the force law given by $f = -k + \mu x^{-2}$.

For (1) we have

$$F = 2Hx + 2kx^{1/2} - (L^2 + \mu) \tag{4.4}$$

$$A = \frac{J}{L}\sin\left[\frac{L}{\sqrt{L^2+\mu}}\sin^{-1}\frac{k\sqrt{x} - (L^2+\mu)}{\sqrt{[k^2 + 2H(L^2+\mu)]x}}\right] \tag{4.5}$$

and for (2) we have

$$F = 2Hx - kx^2 - (L^2 + \mu) \tag{4.6}$$

$$A = \frac{J}{L}\sqrt{x}\sin\left[\frac{L}{2\sqrt{L^2+\mu}}\sin^{-1}\frac{Hx - (L^2 + mu)}{x\sqrt{H^2 - k(L^2+\mu)}}\right]. \tag{4.7}$$

5. Remark

We have shown by explicit demonstration that for motion in any conservative central force field there is always a vector constant of motion, which can at least be expressed in terms of quadratures, on the orbit plane. Although the general solution does not in general yield a simple form expressible in terms of elementary functions, the existence of this general solution does not depend on any condition, such as the orbit must be closed or that there is some hidden higher symmetry. Therefore the mere existence of a vector constant of motion cannot be regarded as any implication that there are some hidden symmetries or that the orbit has some special properties. In fact it has been shown by Fradkin [10] that the dynamical symmetries for all classical central potential problems are O_4 and SU_3.

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